Assignment 1 solutions

1. (a) From what the inhabitants say, we have the following premises.

$$p \leftrightarrow r$$
$$q \leftrightarrow \neg p$$
$$r \leftrightarrow (p \wedge q)$$

We claim that the only possible solution is $(\neg p \wedge q) \wedge \neg r$.

Proof.

1	$p \leftrightarrow r$	premise
2	$q \leftrightarrow \neg p$	premise
3	$r \leftrightarrow (p \wedge q)$	premise
4	r	assumption
5	$(p \wedge q)$	$3, 4,$ special$\leftrightarrow\mathcal{E}$
6	q	$5, \wedge\mathcal{E}$
7	$\neg p$	$2, 6,$ special$\leftrightarrow\mathcal{E}$
8	p	$5, \wedge\mathcal{E}$
9	$\mathbf{F}$	$7, 8, \neg\mathcal{E}$
10	$\neg r$	$4 - 9, \rightarrow\mathcal{I}$
11	$\neg p$	$1, 10,$ special$\leftrightarrow\mathcal{E}$
12	q	$2, 11,$ special$\leftrightarrow\mathcal{E}$
13	$\neg p \wedge q$	$11, 12, \wedge\mathcal{I}$
14	$(\neg p \wedge q) \wedge \neg r$	$10, 13, \wedge\mathcal{I}$

□

(b) From what the inhabitants say, we have the following premises.

$$q \leftrightarrow (p \leftrightarrow \neg p)$$
$$r \leftrightarrow \neg q$$

In this case, there are two possible solutions as show by the following truth table.

p	q	r	$(p \leftrightarrow \neg p)$	$\neg q$	$q \leftrightarrow (p \leftrightarrow \neg p)$	$r \leftrightarrow \not{q}$
T	F	T	F	T	T	T
F	F	T	F	T	T	T

(c) From what the inhabitants say, we have the following premises.

$$\begin{aligned} &p \leftrightarrow \neg(q \vee r) \\ &q \leftrightarrow (p \leftrightarrow r) \\ &r \leftrightarrow (\neg q \wedge \neg r) \end{aligned}$$

We claim that the only possible solution is $(\neg p \wedge q) \wedge \neg r$.

Proof.

1	$p \leftrightarrow \neg(q \vee r)$	premise
2	$q \leftrightarrow (p \leftrightarrow r)$	premise
3	$r \leftrightarrow (\neg q \wedge \neg r)$	premise
4	r	assumption
5	$\neg q \wedge \neg r$	$3, 4$, special$\leftrightarrow\mathcal{E}$
6	$\neg r$	$5, \wedge\mathcal{E}$
7	$\mathbf{F}$	$4, 6, \neg\mathcal{E}$
8	$r \rightarrow \mathbf{F}$	$4-7, \rightarrow\mathcal{I}$
9	$\neg r$	$8, \neg\mathcal{I}$
10	$\neg(\neg q \wedge \neg r)$	$3, 9$, special$\leftrightarrow\mathcal{E}$
11	$\neg q$	assumption
12	$\neg q \wedge \wedge r$	$9, 11, \wedge\mathcal{I}$
13	$\mathbf{F}$	$10, 12, \neg\mathcal{E}$
14	$\neg q \rightarrow \mathbf{F}$	$11-13, \rightarrow\mathcal{I}$
15	$\neg\neg q$	$14, \neg\mathcal{I}$
16	q	$15, \neg\neg\mathcal{E}$
17	$q \vee r$	$16, \vee\mathcal{I}$
18	$\neg p \wedge q$	$16, 18, \wedge\mathcal{I}$
19	$(\neg p \wedge q) \wedge r$	$9, 19, \wedge\mathcal{I}$

$\square$

2. (a) The propositional variables are

p: it is cloudy
q: I have my umbrella
r: it is raining

Correction: There was an error in the solution. What we actually had to prove was

$$\frac{p \to (r \to q)}{(p \to q) \to (p \to r)}$$

In this case, the argument is invalid when p is true, q is true and r is false.

I am leaving the following incorrect solution as an example of a proof using rules of inference. It proves

$$\frac{p \to (r \to q)}{(p \to r) \to (p \to q)}$$

The following proof shows that this argument is valid.

Proof.

1	$p \to (r \to q)$	premise
2	$(p \to r)$	assumption
3	p	assumption
4	r	$2, 3, \to\mathcal{E}$
5	$(r \to q)$	$1, 3, \to\mathcal{E}$
6	q	$4, 5, \to\mathcal{E}$
7	$p \to q$	$3-6, \to\mathcal{I}$
8	$(p \to r) \to (p \to q)$	$2-7, \to\mathcal{I}$

□

(b) The propositional variables are

p: n is prime
q: $n+2$ is prime
r: $n+4$ is prime

so we have to prove

$$\begin{array}{c} p \to q \\ r \to q \\ \hline r \to p \end{array}$$

This argument is invalid. Indeed when p is false, q is true and r is true, both premises are true but the conclusion is false. That is, when n is not prime, $n+2$ is prime and $n+4$ is prime.

Note that the argument would be invalid even if there were no number n for which n is not prime, $n+2$ is prime and $n+4$ is prime (since, for example, we are making no assumptions about what the word "prime" means here).

(c) The propositional variables are

p: I have an account
q: I know my password
r: I can log in

so we have to prove

$$\frac{p}{(r \to q) \vee (q \to r)}$$

Note that here the premise p does not appear in the conclusion. However, this does not mean that the argument is invalid. Indeed, there are valid arguments which need no premise at all and this happens to be one of them.

Here it happens to be useful if we could use a tautology of the form "$P \vee \neg P$" (where we have many possible choices of P which could complete our proof, for example $P = r$ or $P = r \to q$). Of course, we cannot simply state $P \vee \neg P$ just because we "know" that it is a tautology. We have to prove it first. We give a proof by contradiction in this special case.

Proof.

1	p	premise
2	$\neg(r \vee \neg r)$	assumption
3	r	assumption
4	$r \vee \neg r$	$3, \vee\mathcal{I}$
5	$\mathbf{F}$	$2, 4, \neg\mathcal{E}$
6	$r \rightarrow \mathbf{F}$	$3-5, \rightarrow\mathcal{I}$
7	$\neg r$	$6, \neg\mathcal{I}$
8	$r \vee \vee r$	$7, \vee\mathcal{I}$
9	$\mathbf{F}$	$2, 8, \neg\mathcal{E}$
10	$\neg(r \vee \neg r) \rightarrow \mathbf{F}$	$2-9, \rightarrow\mathcal{I}$
11	$r \vee \neg r$	$10, \neg\mathcal{I}$
12	r	assumption
13	q	assumption
14	r	12
15	$q \rightarrow r$	$13-14, \rightarrow\mathcal{I}$
16	$(r \rightarrow q) \vee (q \rightarrow r)$	$15, \vee\mathcal{I}$
17	$r \rightarrow ((r \rightarrow q) \vee (q \rightarrow r))$	$12-16, \rightarrow\mathcal{I}$
18	$\neg r$	assumption
19	r	assumption
20	$\mathbf{F}$	$18, 19, \neg\mathcal{E}$
21	q	$20, \mathbf{F}\mathcal{E}$
22	$r \rightarrow q$	$19-21, \rightarrow\mathcal{I}$
23	$(r \rightarrow q) \vee (q \rightarrow r)$	$22, \vee\mathcal{I}$
24	$\neg r \rightarrow ((r \rightarrow q) \vee (q \rightarrow r))$	$18-23, \rightarrow\mathcal{I}$
25	$(r \rightarrow q) \vee (q \rightarrow r)$	$11, 17, 24, \vee\mathcal{E}$

$\square$

Note that it was also possible to avoid proving $P \vee \neg P$ first by taking the negation of the conlusion as assumption, then showing that $r \rightarrow \mathbf{F}$ which would allow us to use $\neg r$ in the second part of the proof. In that case, at the end, we would simply use $\neg\neg\mathcal{E}$ to get our conclusion.

3. (a) The propositional variables are

p: the gostak distims the doches
q: the gostak is in the delcot

so we have to prove

$$\frac{p}{(q \to p)}$$

Proof.

1	p	premise
2	q	assumption
3	p	1
4	$q \to p$	$2-3, \to\mathcal{I}$

$\square$

(b) The propositional variables are

p: there is a polynomial time algorithm for 3-SAT
q: there is a polynomial time algorithm for 3-colouring

so we have to prove

$$\frac{p \to q}{\neg p \vee q}$$

Proof.

1	$p \to q$	premise
2	$\neg(\neg p \vee q)$	assumption
3	p	assumption
4	q	$1, 3, \to\mathcal{E}$
5	$\neg p \vee q$	$4, \vee\mathcal{I}$
6	$\mathbf{F}$	$2, 5, \neg\mathcal{E}$
7	$p \to \mathbf{F}$	$6, \to\mathcal{I}$
8	$\neg p$	$7, \neg\mathcal{I}$
9	$\neg p \vee q$	$8, \vee\mathcal{I}$
10	$\mathbf{F}$	$2, 9, \neg\mathcal{E}$
11	$\neg((\neg p \vee q) \to \mathbf{F})$	$2-10, \to\mathcal{I}$
12	$\neg\neg((\neg p \vee q) \to \mathbf{F})$	$11, \neg\mathcal{I}$
13	$(\neg p \vee q) \to \mathbf{F}$	$12, \neg\neg\mathcal{E}$

∎

(c) The propositional variables are

p: it is safe to silflay
q: there are hrair hombril outside
r: the hrududu is embleer

so we have to prove

$$\begin{array}{l} \neg q \to p \\ q \to r \\ \neg r \\ \hline p \end{array}$$

Proof.

1	$\neg q \to p$	premise
2	$q \to r$	premise
3	$\neg r$	premise
4	q	assumption
5	r	$2, 4, \to\mathcal{E}$
6	$\mathbf{F}$	$3, 5, \neg\mathcal{E}$
7	$q \to \mathbf{F}$	$4-6, \to\mathcal{I}$
8	$\neg q$	$7, \neg\mathcal{I}$
9	p	$1, 8, \to\mathcal{E}$

$\square$

4. (a) The truth table is

p	q	$\neg(p \wedge q)$	$(\neg p \vee q)$
T	T	F	T
T	F	T	F
F	T	T	T
F	F	T	T

(b)

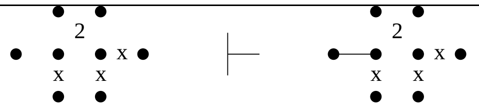
$$\neg p \vee \neg q \vdash \neg(p \wedge q)$$

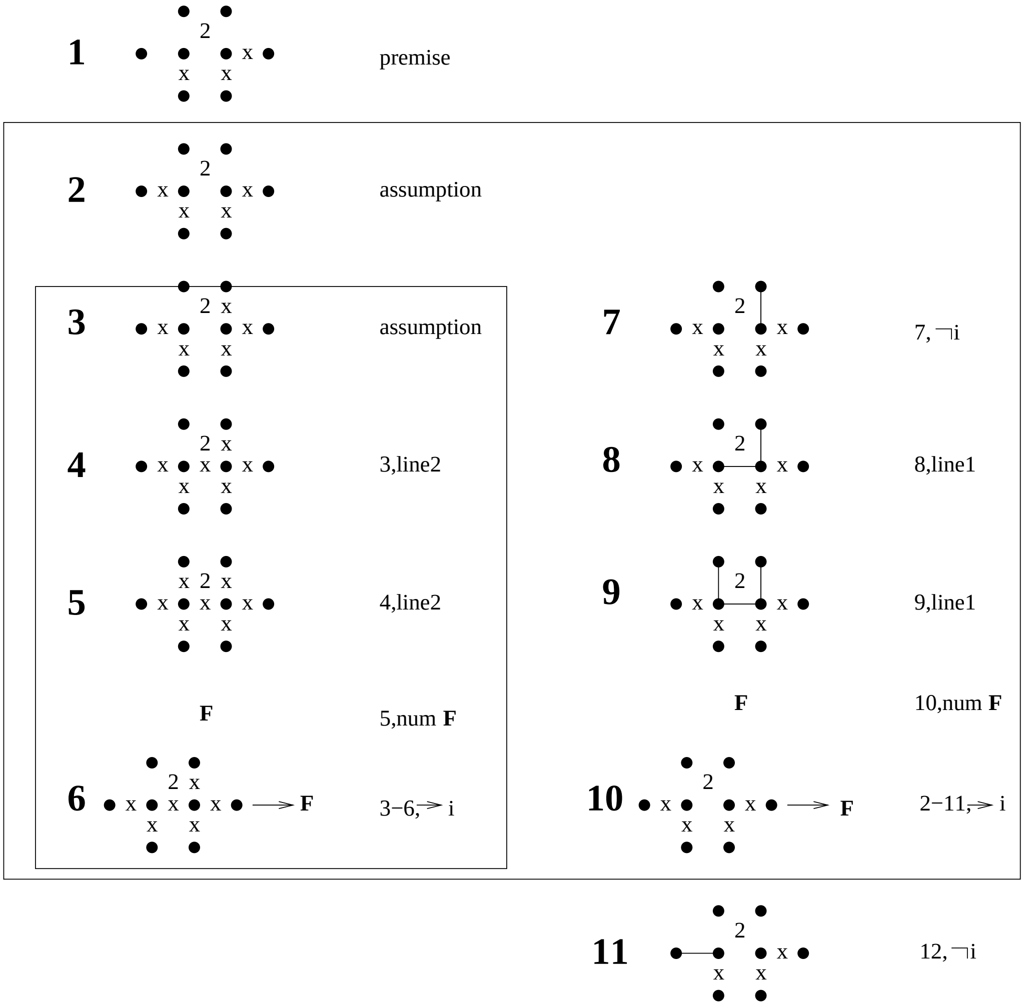
Proof.

1	$\neg p \vee \neg q$	premise
2	$p \wedge q$	assumption
3	$\neg p$	assumption
4	p	$2, \wedge\mathcal{E}$
5	$\mathbf{F}$	$3, 4, \neg\mathcal{E}$
6	$\neg p \rightarrow \mathbf{F}$	$3-5, \rightarrow\mathcal{I}$
7	$\neg q$	assumption
8	q	$2, \wedge\mathcal{E}$
9	$\mathbf{F}$	$7, 8, \neg\mathcal{E}$
10	$\neg q \rightarrow \mathbf{F}$	$7-9, \rightarrow\mathcal{I}$
11	$\mathbf{F}$	$1, 6, 10, \wedge\mathcal{E}$
12	$(p \wedge q) \rightarrow \mathbf{F}$	$2-11, \rightarrow\mathcal{I}$
13	$\neg(p \wedge q)$	$12, \neg\mathcal{I}$

$\square$

5. (a) Here is the proof by contradiction.

Claim: 

Proof: 

(b) See the posted save file for full solution.

It should be noted that move 15 (x at $v(2,1)$) is deduced by contradiction. A line there would lines at the bottom and right in the 2 diagonal to it followed by two non-lines around the 2 diagonal to that 2.

A similar argument is used at move 35 (x at $v(4,2)$) where a line lead to lines around the 2 to the upper right which leads to too many non-lines around the 3 in the upper right corners.

Note that after move 98, the upper right corner of the puzzles cannot be obtained by inference since there are actually 4 possible solutions.

6.

Theorem 1.

$$P_1, \ldots, P_k \vdash Q \Rightarrow P_1, \ldots, P_k \models Q$$

Proof. Suppose there exists $P_1, \ldots, P_k, Q$ such that $P_1, \ldots, P_k \vdash Q$ but $P_1, \ldots, P_k \not\models Q$. Choose $P_1, \ldots, P_k, Q$ with this property which minimizes the length of (the shortest) proof of $P_1, \ldots, P_k \vdash Q$.

If Q is exactly P_i for some i then $P_1, \ldots, P_k \models Q$ since the columns for P_i and Q are exactly the same so whenever P_i is true, so is Q. Contradiction to $P_1, \ldots, P_k \not\models Q$.

Otherwise, consider the last inference rule used in a shortest proof $\mathcal{P}$ of $P_1, \ldots, P_k \vdash Q$.

We claim that $P_1, \ldots, P_k \models Q'$ for any Q' which is not boxed and not the last line of the proof $\mathcal{P}$. Indeed, otherwise, $P_1, \ldots, P_k \not\models Q'$ but $P_1, \ldots, P_k \vdash Q'$ and it has a shorter proof (obtained by deleting some lines of $\mathcal{P}$). This contradicts our choice of $P_1, \ldots, P_k, Q$.

Suppose the last rule used in $\mathcal{P}$ is not $\wedge\mathcal{E}$. By the previous claim, whenever $P_1, \ldots, P_k$ are all true (in the truth table), $P \wedge Q$ is true (in the truth table) and the following truth table shows that P is true whenever $P \wedge Q$ is true.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

Therefore, we have shown that $P_1, \ldots, P_k \models Q$ in this case. This is a contradiction to our initial assumption. So the last rule used is not $\wedge\mathcal{E}$.

Similarly, the following truth tables show that the last rule used in $\mathcal{P}$ is not $\rightarrow\mathcal{I}$.

P	Q	**F**	$P \wedge Q$	$P \rightarrow Q$	$P \rightarrow F$	$\neg P$	$\neg\neg P$
T	T	F	T	T	F	F	T
T	F	F	F	F	F	F	T
F	T	F	F	T	T	T	F
F	F	F	F	T	T	T	F

P	Q	R	$P \vee Q$	$P \rightarrow R$	$Q \rightarrow R$
T	T	T	T	T	T
T	F	T	T	T	T
F	T	T	T	T	T
F	F	T	F	T	T
T	T	F	T	F	F
T	F	F	T	F	T
F	T	F	T	T	F
F	F	F	F	T	T

Therefore, we have shown that the last rule must be $\to\mathcal{I}$. This implies that there was some box before the last line which started at some assumption A and arrived at some conclusion C.

We claim that $P_1, \ldots, P_k, A \models C$. Notice that if we remove the box around A, move the line containing A in $\mathcal{P}$ to the $k+1$st line and take A as premise, all the steps in our proof are still correct. Now if we remove the last line (which contains Q), we have obtained a proof of $P_1, \ldots, P_k, A \models C$ which is shorter than $\mathcal{P}$. Therefore, if $P_1, \ldots, P_k, A \not\models C$, we have obtained a contradicts our choice of $P_1, \ldots, P_k$.

Therefore, by looking at the truth table containing $P_1, \ldots, P_k, A, C$, we see that whenever $P_1, \ldots, P_k, A$ are true, C is true. So if we add a column $A \to C$ to this table, it will be true whenever $P_1, \ldots, P_k$ is true.

Therefore, we have shown that $P_1, \ldots, P_k \models Q$ in all cases. This is a contradiction to our initial assumption. $\square$