Assignment 4 solutions

1. (a) This statement is false.

   Let $S_1 = \{0, 1\}, S_2 = \{1, 2\}, S_3 = \{0, 2\}$. Then $S_1 \cap S_2 = \{1\}$, $S_2 \cap S_3 = \{2\}$ and $S_1 \cap s_3 = \{0\}$ which are all non-empty but S_1 is missing 2, S_2 is missing 0 and S_3 is missing 1 so no element is in all three sets.

   (b) *Proof.* If T contains only one vertex v then the statement is true since all subtrees contain v (since the empty graph has no vertex is common with any other subtree and if there is only one subtree T_1, the statement is true).

   Now suppose the statement is true for all trees T of size $n - 1$ for some $n \geq 2$. Let T be any tree of size n and $T_1, \ldots, T_k$ be subtrees of T that pairwise intersect ($V(T_i) \cap V(T_j) \neq \emptyset \forall i, j$).

   By the lemma proven in class, T contains a vertex v of degree 1. Either $\exists j$ such that $V(T_j) = \{v\}$ or all subtrees T_i containing v also contains at least one other vertex.

   In the first case, since every subtree intersects T_j, so $v \in \cap_{i=1}^{k} V(T_i)$ which proves the statement (for T).

   In the second case, note that u, the only neighbour of v, is on the path from v to any other vertex in T. Since every subtree is a tree and therefore connected, all subtrees T_i containing v also contains u.

   We claim that $T_1 - v, T_2 - v, \ldots, T_k - v$ pairwise intersect (and are all subtrees of $T - v$). Here, we use the convention that $T_i - v = T_i$ if $v \notin T_i$. Indeed, if not, there is a pair of subtrees T_ℓ and T_m that only intersected in v (since the subtrees pairwise intersect initially). But then $v \in T_\ell$ so $u \in T_\ell$ and $v \in T_m$ so $u \in T_m$. Therefore, $u \in (T_\ell - v) \cap (T_m - v)$ which is a contradiction.

   Thus, by our hypothesis, $T_1 - v, \ldots, T_k - v$ (as subtrees of $T - v$) contain a vertex in their intersection (we have already proven in class that $T - v$ is a tree if T is a tree and v has degree 1).

   But $\cap_{i=1}^{k} V(T_i) - v$ is contained in $\cap_{i=1}^{k} V(T_i)$ so $\cap_{i=1}^{k} V(T_i)$ also contains a common vertex.

   Thus, in all cases, we have shown that the statement holds for T. Since T is an arbitrary tree of size n, we have proven the statement by induction. □

2. (a) Suppose the statement is false. Then for some i and j, there is a path $Q = q_1 = p_i, q_2, \ldots, q_\ell = p_j$ which is of lower weight than $p_i, \ldots, p_j$. But now we can replace the subpath from p_i to p_j by Q. The weight of $p_1, p_2, \ldots, p_i, q_2, \ldots, q_{\ell-1}, p_j, p_{j+1}, \ldots, p_k$ is

$$\begin{aligned} & \sum_{m=1}^{i-1} w_{p_m, p_{m+1}} + \sum_{m=1}^{\ell} w_{q_m, q_{m+1}} + \sum_{m=j}^{k-1} w_{p_m, p_{m+1}} \\ < \; & \sum_{m=1}^{i-1} w_{p_m, p_{m+1}} + \sum_{m=i}^{j-1} w_{p_m, p_{m+1}} + \sum_{m=j}^{k-1} w_{p_m, p_{m+1}} \end{aligned}$$

   But this is the weight of P. This gives a contradiction since we have found a path from p_1 to p_k of lower weight than P (which is suppose to be a minimum weight path).

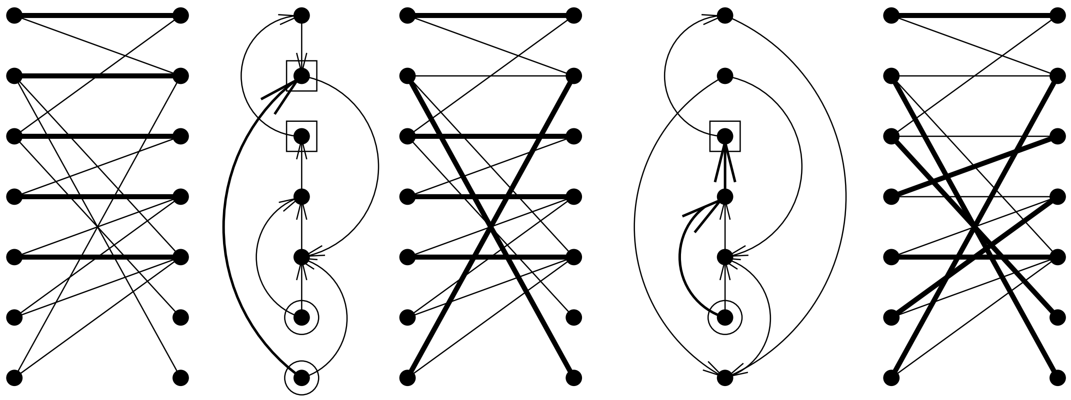
   (b) Suppose F is the set of chosen edges for some input G and weights w.

   First we claim that (V, F) contains no cycles. If this claim is false, consider the first iteration where the set of chosen edges F_i yields a cycle in (V, F_i). Let e be the last chosen edge. Now at the beginning of every iteration, S contains all vertices incident to a chosen edge (since whenever we chose an edges, we add its endpoint not in S to S). So e had both endpoints in S which is a contradiction (we need to chose an edge from S to $V - S$).

   Next we claim that (V, F) is connected. This is true since every vertex has a path to s (by following prev pointers) and concatenating the path from u to s and the path from s to v gives a walk from u to v for any u and v.

Therefore (V, F) is a tree.

3. (a) Here is a run of the second algorithm we saw.



From left to right, the figures are:

- the input graph with the initial matching,
- the digraph H that we built based on the initial matching where circles are unmatched vertices in A and squares are neighbours of unmatched vertices in B and the highlighted edges from a path from a vertex of A to a neighbour of B,
- the new matching after swapping edges on the augmenting path we found,
- the digraph H that we built based on the new matching, and
- the final matching after swapping on the second augmenting path we found.

(b) (see external figure)

From left to right, the first row contains

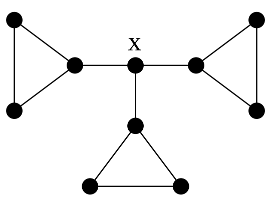
- the input graph,
- the shortest path from a to all other vertices,
- the shortest path from e to all other vertices,
- the shortest path from f to all other vertices, and
- the shortest path from g to all other vertices.

Note that we only needed any 3 of the four shortest path trees.

From left to right, the first row contains

- the graph H we built,
- a matching of value 22,
- a matching of value 21 (which is the minimum weight maximum matching),
- a matching of value 22, and
- the set of edges to be double in the input graph.

(c) This statement is false. Here is a counter-example.



This graph satisfies the hypothesis but does not have a perfect matching.

It is easy to see that this graph contains no perfect matching. In a perfect matching, x is matched to exactly one vertex so two of the triangles do not contain a vertex matched to x. Then, one of these two triangles will contain an unmatched vertex. This is a contradiction to the fact we had a perfect matching.

It is also easy to check that $|N(S)| \geq |S|$ for all subsets of vertices by first checking that this is true for a triangle (without counting their edge to x). Thus, this property is satisfied for the graph without x. Then, we just need to check that the property still holds when we add x.

4. (a) We use the shorthand $\prod_{i=1}^{k} a_i$ to denote the product $a_1 a_2 \ldots, a_k$. Let $T = S_1 \times S_2 \times \ldots \times S_k$. Without loss of generality, for each i, $S_i = \{0, 1, \ldots, |S_i| - 1\}$ (if not, order the elements of S_i and we apply the following argument to the index of the elements).

We define $f : T \to \{0, 1 \ldots, \prod_{i=1}^{k} |S_i| - 1\}$ as follows.

$$f((s_1, s_2, \ldots, s_k)) = s_k + |S_k| s_{k-1} + |S_k||S_{k-1}| s_{k-2} + \ldots + s_1 \prod_{i=2}^{k} |S_i| = \sum_{i=1}^{k} s_i \prod_{j=i+1}^{k} |S_j|$$

We now need to show that f is a bijection.

First, we show that f is injective. Suppose that $f((s_1, \ldots, s_k)) = f((t_1, \ldots, t_k))$ but $(s_1, \ldots, s_k)$ and $(t_1, \ldots, t_k)$ differ. Let ℓ be the minimum index in which they differ (i.e., $s_\ell \neq t_\ell$ but $s_j = t_j$ for all $j < \ell$). Without loss of generality, $s_\ell < t_\ell$ (otherwise, we can switch the names of $(s_1, \ldots, s_k)$ and$(t_1, \ldots, t_k)$). Then we can rewrite the equalities

$$\begin{aligned}
f((s_1, \ldots, s_k)) &= f((t_1, \ldots, t_k)) \\
\sum_{i=1}^{k} s_i \prod_{j=i+1}^{k} |S_j| &= \sum_{i=1}^{k} t_i \prod_{j=i+1}^{k} |S_j| \\
\sum_{i=\ell}^{k} s_i \prod_{j=i+1}^{k} |S_j| &= \sum_{i=\ell}^{k} s_i \prod_{j=i+1}^{k} |S_j| \\
(s_\ell - t_\ell) \prod_{j=\ell+1}^{k} |S_j| + \sum_{i=\ell+1}^{k} s_i \prod_{j=i+1}^{k} |S_j| &= \sum_{i=\ell+1}^{k} s_i \prod_{j=i+1}^{k} |S_j|
\end{aligned}$$

But now we claim that the left hand side is negative while the right hand side is clearly non-negative (the sum of non-negative numbers is non-negative).

Indeed, since $s_\ell < t_\ell$, $s_\ell - t_\ell < 0$ and

$$(s_\ell - t_\ell) \prod_{j=\ell+1}^{k} |S_j| \leq - \prod_{j=\ell+1}^{k} |S_j|$$

but each $s_i < |S_i|$ so

$$\sum_{i=\ell+1}^{k} s_i \prod_{j=i+1}^{k} |S_j| < \sum_{i=\ell+1}^{k} \prod_{j=i}^{k} |S_j|$$

Therefore their sum is negative.

Second, we show that f is surjective.

Let $x \in \{0, 1 \ldots, \prod_{i=1}^{k} |S_i| - 1\}$. We need to show that there exists $(s_1, \ldots, s_k) \in T$ such that $f((s_1, \ldots, s_k)) = x$.

This is true when

$$s_j = \lfloor \frac{x}{\prod_{i=j+1}^{k} |S_i| - 1} \rfloor \mod |S_j|$$

To see this we can compare the values of $f((s_1,\ldots,s_k))$ and x taken modulo $\prod_{i=2}^{k} |S_i|$. We see that both are s_1 (by our definition of s_1). Therefore, we can subtract s_1 and divide by $|S_2|$ from both and repeatedly compare them (the second time we would look at the remainder when divided by $\prod_{i=3}^{k} |S_i|$). This shows that we indeed have $f((s_1,\ldots,s_k)) = x$.

Therefore, f is a bijection. So $|T| = |\{0,1,\ldots,\prod_{i=1}^{k} |S_i| - 1\} = \prod_{i=1}^{k} |S_i|$.

(b) Without loss of generality, $B = \{0,1,\ldots,|B|-1\}$ (if not, order the elements of B and we apply the following argument to the index of the elements).

Let $S = B \times B \times \ldots \times B$, the Cartesian product of $|A|$ copies of B.

Let T be the set of all functions from A to B.

We define $g : T \to S$ as follows.

Let $f \in T$. Then f assigns to each element of A an element of B. Let $A = \{a_1, a_2, \ldots, a_x$ (where $x = |A|$). Then $(f(a_1), f(a_2), \ldots, f(a_x))$ is an element of S. We set

$$g(f) = (f(a_1), f(a_2), \ldots, f(a_x))$$

First, we show that g is injective. Suppose $g(f_1) = g(f_2)$. Then, by our definition of g,

$$(f_1(a_1), f_1(a_2), \ldots, f_1(a_x)) = (f_2(a_1), f_2(a_2), \ldots, f_2(a_x))$$

and by the definition of Cartesian product, $f_1(a_i) = f_2(a_i)$ for all i. Thus, $f_1 = f_2$ (as functions).

Second, we show that g is surjective. Let $(b_1, b_2, \ldots, b_x) \in S$ (for all i, $b_i \in B$ but they need not be distinct). Then we can define a function $f : A \to B$ as $f(a_i) = b_i$ for i from 1 to x. f is indeed a function since it assigns an element of B (namely b_i) to each element of A. And, by definition of g, $g(f) = (b_1, b_2, \ldots, b_x)$.

Therefore g is a bijection. So $|T| = |S|$ and by question a), $|S| = |B|^{|A|}$.

5. This is known as König's theorem.

Let $G = (V, E)$ be any bipartite graph (with parts A and B). Let M be a maximum matching in G and X be a minimum size vertex cover in G.

First, we see that $|M| \leq |X|$ since for each edge in M, X contains one of its endpoints (by definition of a vertex cover) and edges in M do not share any endpoints (by definition of a matching).

Second, we claim that $|M| \geq |X|$.

Let G_1 be the subgraph of G with vertex set $(X \cap A) \cup (N(X \cap A) - X)$ (i.e., all vertices of the cover in A and their neighbours not in the cover) and all edges between these vertices (so G_1 is an induced subgraph). We claim that G_1 contains a matching of size $|X \cap A|$.

Note that if some subset S of $X \cap A$ has $|N(S)| < |S|$ in G_1 then we can replace S by $N(S)$ in G to obtain a smaller vertex cover of G (we only need to check that all edges incident to a vertex of S is covered). This is a contradiction to the minimality of X.

Thus, for all $S \subseteq X \cap A$, $|N_{G_1}(S)| \geq |S|$. Thus, by Hall's theorem (applied to G_1 where we add $|N(X \cap A)| - |X \cap A|$ dummy vertices to A incident to all vertices in $N(X \cap A)$), G_1 contains a matching of size $|X \cap A|$.

Similarly, we can define G_2 to be the subgraph of G with vertex set $(X \cap B) \cup (N(X \cap B) - X)$ and all edges between these vertices. Then using the same argument (switching A and B), we get that G_2 contains a matching of size $|X \cap B|$.

Since G_1 and G_2 are subgraphs of G with no vertices in common, G contains a matching of size $|X \cap A| + |X \cap B = |X|$. This is a lower bound on $|M|$ by maximality of M.