Assignment 5

This assignment is worth 4% of your final grade. Answer all questions from 1 to 4. You may also answer the bonus questions for extra marks but you cannot receive partial marks for the bonus questions.

Please show steps of your reasoning. You may use any theorems (or lemmas or claims) that we proved in class if you clearly state which theorem you are using. Note that if a theorem you want to use is similar to (but not exactly) one seen in class, you should prove it first. Do not simply state that the proof is similar (write down the whole proof).

1. (a) (6 points)

   Definition 1. A graph G is *d-regular* if all vertices of G have degree d.

   Prove the following statement.

   For any d, any bipartite d-regular graph G has a perfect matching.

   (b) (4 points)

   If G be a bipartite d-regular graph then G contains d perfect matchings $M_1, M_2, \ldots, M_d$ such that $M_i \cap M_j = \emptyset$ for all $i \neq j$ (i.e., G contains d edge disjoint matchings).

2. For each of the following questions, you need to show your steps but you need not give a formal proof that your answer is correct. Write your final answer in a closed form formula. You may optionally evaluate it (but do not only leave an evaluated value).

   (a) (2 points)

   In how many ways can we re-order a deck of 52 cards so that no cards is at its original position. e.g., if 3♣ is originally the 10th cards from the top, it is not the 10th cards from the top after re-ordering.

   (b) (2 points)

   How many different hands of 10 cards can you draw from a standard deck of 52 cards if only suits matter. e.g., $\{J\heartsuit, K\heartsuit, 9\heartsuit, 7\heartsuit, 10\diamondsuit, A\diamondsuit, 2\spadesuit, 4\clubsuit, 7\clubsuit, 9\clubsuit, 10\clubsuit\}$ and $\{9\heartsuit, 3\heartsuit, 2\diamondsuit, 8\spadesuit, Q\clubsuit, 5\clubsuit, 6\heartsuit, 10\clubsuit, 3\diamondsuit, 9\clubsuit, 5\heartsuit\}$ are considered the same hand.

   (c) (2 points)

   How many different hands of 5 cards can you draw with no ace, no face card and no ♣ from a non-standard deck of 24 cards consisting only of 9,10,J,Q,K,A?

   (d) (4 points)

   How many different hands of 5 cards can you draw from a standard deck of 52 cards where the hand contains at least 3 cards of the same suit?

3. Suppose you shuffle a deck of n distinct cards in the following way.

   - Split the deck into 2 piles: the first k cards and the last $n - k$ cards.
   - Put one pile into your left hand and the other pile into your right hand.
   - Construct a new deck by alternating between a card in your left hand and a card in the right hand.

   e.g., suppose the initial deck is $1, 2, 3, 4, 5, 6, 7, 8, 9, 10$ (so $n = 10$). If you split it into piles $1, 2, 3$ and $4, 5, 6, 7, 8, 9, 10$ (so here $k = 3$) and put $1, 2, 3$ into your left hand then the new deck obtained is $1, 4, 2, 5, 3, 6, 7, 8, 9, 10$.

   (a) (4 points)

   How many different decks can you obtain by shuffling once? Remember to show that the decks you are counting are actually different.

(b) (3 points)

Prove the following statement.

Let G be a graph where all vertices have degree at most Δ. For any $v \in V(G)$ and any integer d, the number of vertices whose (shortest path) distance from v is at most d is at most Δ^{d+1} (here all edges have weight 1). i.e., show that $\forall v, \forall d, |\{u | d(u,v) \leq d\}| \leq \Delta^{d+1}$.

(c) (3 points)

Use a) to show that in a deck of 52 cards, even after 33 consecutive shuffles, there are decks which cannot be obtained.

4. (a) (5 points)

   How many perfect matchings are there in the complete graph on $2n$ vertices? Two matchings are considered to be different if they differ in at least one edge.

   Prove that your formula is correct.

   (b) (5 points)

   Prove that in any subset of size 101 of $\{1, 2, 3, \ldots, 200\}$, there is always a number which is a multiple of another.

5. **Bonus** (10 points)

   Prove the following statement.

   For any r, there exists an N such that if the set of integers $\{1, 2, \ldots, N\}$ is split into two sets S and T (so that $S \cup T = \{1, 2 \ldots, n\}$ and $S \cap T = \emptyset$) then there exists integers a and d such that either S or T contains the subset $\{a, a+d, a+2d, a+3d, \ldots, a+rd\}$.