Assignment 5 solutions

1. (a) Let G be a d-regular bipartite graph with parts A and B.
   We intend to use Hall's theorem show that G has a perfect matching. Thus, we need to check
   - $|A| = |B|$, and
   - for any set of vertices $S \subseteq A$, $|S| \leq |N(S)|$.

   By lemma seen in class,
   $$\sum_{v \in A} \deg(v) = |E(G)| = \sum_{v \in B} \deg(v)$$
   and in our case, $\deg(v) = d$ for any vertex v. Therefore, $d|A| = d|B|$ and $|A| = |B|$.
   Now suppose $S \subseteq A$ but $|S| > |N(S)|$. Let H be the subgraph with vertices $S \cup N(S)$ and all edges of G between them. Then, by the same lemma, the number of edges in H is $d|S|$ (when counting from the part containing S). Since H is a subgraph of G, when counting degree from $N(S)$, we get that the number of edges of H is at most $d|N(S)| < d|S|$. This is a contradiction to the lemma (i.e., we get a different count when counting from different sides).
   Therefore, by Hall's theorem, G contains a perfect matching.

   (b) We prove this statement by induction on d.
   If G is a 1-regular graph, by a), G contains a perfect matching M_1. This is 1 edge disjoint matching. This proves the statement for $d = 1$.
   Suppose the theorem is true for $d - 1$ and G is a d-regular graph. By a), G contains a perfect matching M_d. Since every vertex of G is incident to exactly one edge of M_d, the subgraph $H = (V(G), E(G) \setminus M_d)$ of G is $d-1$-regular. By induction, H contains $d-1$ edge disjoint perfect matchings $M_1, M_2, \ldots, M_{d-1}$. M_d is edge disjoint from all these matchings since the edges in M_d do not appear in H. Since H contains the same vertices as G, $M_1, M_2, \ldots, M_{d-1}$ are perfect matchings in G as well.
   Hence, $M_1, M_2, \ldots, M_{d-1}, M_d$ consist of d edge disjoint matchings in G.

2. (a) This is simply the number of derangement from the set of 52 cards to itself. We have seen in class that this is
   $$\sum_{i=0}^{52} (-1)^i \frac{52!}{i!}$$

   (b) Note that we have more than 10 cards of any given suit so this is just the number of combinations where repetition is allowed. Here, we are chosen 10 elements (cards) from a set of 4 elements (suits)
   $$\binom{10+4-1}{10} = \binom{13}{10}$$

   (c) The number of cards which are neither an ace, a face or a ♣ is 6 (namely, the 9 and 10 of non-♣ suits). We have to pick 5 cards out of these so the number of different hands is $\binom{6}{5} = 6$.

   (d) A hand with exactly 3 cards of ♣ can be obtained by first selecting 3 ♣ cards and then selecting 2 non-♣ cards. This is $\binom{13}{3}\binom{39}{2}$ possibilities.
   Similarly the number of hands with exactly 4 cards of ♣ is $\binom{13}{4}\binom{39}{1}$.
   Finally, the number of hands with exactly 4 cards of ♣ is $\binom{13}{5}$.
   There are 4 choices of suit instead of ♣.
   Note that if we draw 5 cards, it is not possible to get two suits where we have at least 3 cards with both suits. We would normally have to use inclusion-exclusion here but it happens that the intersections are all empty.
   $$4\left(\binom{13}{3}\binom{39}{2} + \binom{13}{4}\binom{39}{1} + \binom{13}{5}\right)$$

3. (a) There are k choices where to split the deck.

   For any split, there are two choices for which pile to put into which hand.

   This seems to be $2n$ different decks at first. However, there are choices which lead to the same final deck. For example, if we chose $k = 0$, we get the original deck back no matter which "deck" we put in which hand (in fact one deck is empty).

   But actually, this is essentially the only case where we can get the same deck. Indeed, suppose we split the deck before the kth card from some k between 2 and n. The top two cards of the new deck after shuffling is either c_1, c_k or c_k, c_1 where c_i is the ith card from the top in the original deck. Thus, no two decks from a shuffle has the same two top cards.

   Actually, we obtain the deck in the same order when $k = 2$ and we put the single card into our left hand. Thus, we need not count the case $k = 1$ and $k = n$ since it is counted when $k = 2$.

   So the total number of different decks obtainable via a single shuffle is $2(n - 1)$.

   (b) First, we prove that the number of neighbours at distance exactly d is at most Δ^d. Then it will just be a matter of bounding the geometric series.

   Lemma 1. *Let G be a graph where all vertices have degree at most Δ. For any vertex $v \in V(G)$ and any d, the number of vertices at distance d from v is at most Δ^d.*

   Proof. This is true for $d = 0$ since there is only one vertex at distance 0 from v, namely v itself.

   Suppose the statement is true for distance $d - 1$. Then note that all vertices at d from v are adjacent to some vertex at distance $d - 1$ from v. Thus, there are at most Δ times the number of vertices at distance $d - 1$. So by induction, this number is $\Delta^{d-1}\Delta = \Delta^d$.

   We have thus proven the statement by induction. □

   Now, by the previous lemma, the number of vertices at distance at most d is bounded above by

$$\sum_{i=0}^{d} \Delta^i = \frac{1 - \Delta^{d+1}}{1 - \Delta} = \frac{\Delta^{d+1} - 1}{\Delta - 1} \leq \Delta^d$$

   (c) Consider the graph G where vertices are all 52! decks and there is an edge between two vertices if and only if one deck can be obtained from the other through a single shuffle.

   The vertices of G have degree $2 * (52 - 1) = 102$. By b), the number of decks which can be reach within 33 shuffle is therefore

$$102^{33}$$

   which is less than 52 factorial. Thus, there exists unobtainable decks after 33 shuffles.

4. (a)

   Claim 1. *There are*

$$(2n - 1)(2n - 3) \dots 1 = \prod_{i=0}^{n-1} (2i + 1)$$

   perfect matchings in K_{2n}.

   Proof. Label the vertices of K_{2n} by $1, 2, \dots, 2n$.

   To obtain a matching for K_{2n}, we can repeatedly choose a vertex to match to the smallest labelled remaining vertex.

   For example, at the beginning, we decide what to match to 1. If we matched 1 to 3, we would then decide what to match to 2 (the lowest labelled remaining unmatched vertex). If we matched 2 to 5 then we would then decide what to matched to 4 (since 3 is already matched to 1). And so on.

We have $n-1$ choices for our first choice. We always have $n-3$ choices for our second choice. And so on.

Now it is a matter of showing that we cannot obtain the same matching through different choices and that we can obtain every matching in this way.

Suppose we make different choices. Look at the first time where the choices differ. By making different choices, whatever vertex was the lowest index at that point got matched to different vertices (depending on which decision we made). Once we match a vertex, we never change our mind again. This is a Thus, if we make different choices, we necessarily obtain different matchings.

Given a perfect matching M we would like to obtain from our set of choices, we can look at what 1 is matched to in M. We choose that vertex as our first choice. Then we look at what the lowest labelled remaining vertex is matched to in M. We chose that vertex as our second choice. And so on. This gives us a set of choices which will give us M. $\square$

(b) Let $S = \{s_1, \ldots, s_{101}\}$ be a subset of $\{1, 2, \ldots, 200\}$.

The idea is that for any number x below 100, there is a multiple of x which is between 100 and 200. Then only one of x and this multiple of x can be in S. But we have a subset of size greater than $|S|/2$. However, this idea does not quite work since the multiple of two different numbers could be the same. For example, we could have 51 as a multiple of 2 and as a multiple of 3 as well. However, this problem does not occur if we consider only multiples of x which are powers of 2 (i.e., double x until we are between 100 and 200). This is multiples of powers of 2 of x are also multiples of each other (i.e., $2^i x$ divides $2^j x$ if $i < j$).

We now give a formal proof.

Proof. Let $S = \{s_1, \ldots, s_{101}\}$ be a subset of $\{1, 2, \ldots, 200\}$.

Each number $s_j \in S$ (and in fact any positive integer) can be factored as $s_j = 2^{i_j} x_j$ where x_j is odd. Since every number in S is between 1 and 200, there are 100 possible different values of x_j (namely, all odd numbers between 1 and 200).

By the pigeonhole principle, there are 2 elements s_a and s_b in S with $x_a = x_b$. If $i_a < i_b$ then s_a multiplied by $2^{i_b - i_a}$ is s_b. Otherwise, $i_a > i_b$ and s_b multiplied by $2^{i_a - i_b}$ is s_a.

Thus, we have found a number which is a multiple of another in S. $\square$