Assignment 6

1. Ramsey numbers can be thought of as in a different way.

   Definition 1. $R(s,t)$ is the smallest number such that in any colouring of the edges of the complete graph on $R(s,t)$ vertices with two colours red and blue, there is always either

   - a red clique of size s (that is, all edges between vertices of the clique are red), or
   - a blue clique of size t.

   This is an equivalent definition since we can think of the edges of a graph on $R(s,t)$ vertices as being coloured red and the non-edges as being coloured blue.

   This second definition allows for the following generalization.

   Definition 2. $R(r,s,t)$ is the smallest number such that in any colouring of the edges of the complete graph on $R(r,s,t)$ vertices with three colours red, blue and green, there is always either

   - a green clique of size r, or
   - a red clique of size s, or
   - a blue clique of size t.

   As in the case for $R(s,t)$, it is not clear that these numbers even exist.

   In this question, you are asked to prove that they do exists.

   That is, prove some upper bound on $R(r,s,t)$.

   [Hint: It may be useful to prove $R(r,s,t) \leq R(r, R(s,t))$.]

2. (a) (Discrete mathematics elementary and beyond, p.66-67)
   A staircase has n steps. You walk up taking one or two steps at a time. How many ways can you go up. Given a closed form formula as a function of n.
   [Hint: Write the number of ways as a linear recurrence.]

   (b) Solve the linear recurrence

$$\begin{aligned} a_{n+2} &= \frac{a_{n+1} + a_n}{2} \\ a_0 &= 1 \\ a_1 &= 4 \end{aligned}$$

3. (a) (Rosen, p.414, ex 7e))
   Suppose we randomly order the integers 1,2,3,4 where each ordering is equally likely.
   What is the probability that 4 precedes 3 and 2 precedes 1 in the ordering?

   (b) (Rosen, p.400, ex 40)
   Suppose in the Monty Hall problem, there are four doors instead of three. The host still opens one unchosen door with no prize.
   What is the probability of winning by switching? What is the probability of winning without switching?

4. (a) (Rosen, p.415, ex 13)
   Prove $Pr[E \cap F] \geq Pr[E] + Pr[F] - 1$.

(b) (Rosen, p.415, ex 14)
Prove the following generalization of a).

$$Pr[E_1 \cap E_2 \cap \ldots \cap E_n] \geq Pr[E_1] + Pr[E_2] + \ldots + Pr[E_n] - (n-1)$$

(c) (Rosen, p.413 Theorem 3)
Let $n < 2^{k/2}$. Suppose we randomly choose a labelled graph G on n vertices where each labelled graph is chosen with equal probability.

It can be shown that each edge occurs with probability 1/2.

Show that the probability that the graph G we picked contains a stable set of size k or an independent set of size k is at least

$$2\binom{n}{k}\left(\frac{1}{2}\right)^{\binom{k}{2}}$$

5.

Definition 3. Let § be a set of graphs (usually called a *class* of graphs). § is said to be *d-degenerate* if every graph $G \in$ § is either

- contains no vertex, or
- contains a vertex v with $\deg(v) \leq d$ and $G - v \in$ §.

Prove the following statement.

Let § be a set of graphs. If § is d-degenerate and $G \in$ § then G is $d+1$-colourable.