Note on counting

Definition 1. A *set* is a (unordered) collection of distinct elements.

The above is not exactly a definition in the usual sense since we do not know what a collection is. It is rather an emphasis on the fact that sets are only allowed to contain distinct elements.

In this course, we will only work with finite sets.

Definition 2. A *function* f from a set A to a set B, denoted $f : A \to B$, is an assignment of one element of B to each element of A.

$f(a)$ is the element of B assigned to $a \in A$.

Note. We can represent a function (between finite sets) as a bipartite graph. This graph G has vertex set $A \cup B$ and edge set $\{(a, f(a))|a \in A\}$.

Any graph that we build from a function in such a way has only vertices of degree 1 in A.

Although, normally, we do not think of functions as bipartite graphs.

Definition 3. A function $f : A \to B$ is said to be *injective* (or *one-to-one*) if $\forall a_1, a_2 \in A, f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ (i.e., no two elements of A get assigned the same element of B).

A function $f : A \to B$ is said to be *surjective* (or *onto*) if $\forall b \in B \exists a \in A$, such that $f(a) = b$ (i.e., all elements of B are assigned some element of A).

A function is *bijective* if it is both injective and surjective.

A bijective function is called a *bijection*.

Note. In terms of graphs, a function is injective if the graph we build has $\deg(b) \leq 1 \forall b \in B$ and a function is surjective if $\deg(b) \geq 1 \forall b \in B$. Thus, a function is bijective if all vertices have degree 1 in the graph we build. So they look like a graph with a single perfect matching.

Bijections helps us count because of the following easy theorem.

Theorem 1. *If there is a bijection between A and B then $|A| = |B|$..*

Usually, we find bijections between a set whose size is unknown and a set whose size is known.

Example 1.

Claim 1. *The number of subsets of a set S of size n is 2^n.*

The idea of the proof is just to think of subsets as binary representions of a number. The ith bit of that number is 1 if the i element of S is in the subset.

This suggests that we should order the elements in S so that we can refer to the ith element. Since S is finite, we just let $S = \{s_0, s_2, \ldots, s_{n-1}\}$ (so that we can now refer to the "labels" s_i rather than directly to the elements of S themselves).

Now we can define a function $f : \{ \text{ subsets of } S\} \to \{0, 1, \ldots, 2^n - 1\}$ as

$$f(T) = \sum_{s_i \in T} 2^i$$

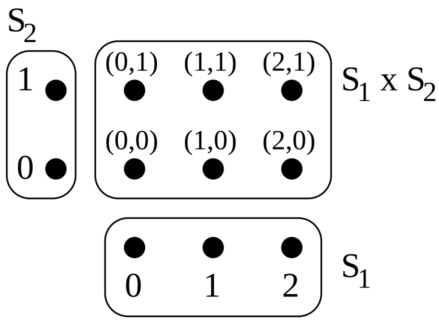
Then it is just a matter of checking that f is indeed a bijection (we will not do this here).

For example, if $S = \{a, b, c\}$ and we labelled the elements of S as $s_0 = a, s_1 = b, s_2 = c$, then our function is

$$\begin{aligned} f(\emptyset) &= 0 \\ f(\{a\}) &= 1 \\ f(\{b\}) &= 2 \\ f(\{b,a\}) &= 3 \\ f(\{c\}) &= 4 \\ f(\{c,a\}) &= 5 \\ f(\{c,b\}) &= 6 \\ f(\{c,b,a\}) &= 7 \end{aligned}$$

Definition 4. Let S_1, S_2 be two sets. Then $T = \{(s_1, s_2) | s_1 \in S_1, s_2 \in S_2\}$ is called the *Cartesian product* of S_1 and S_2. It is denote by $S_1 \times S_2$.

Example 2. Here is an illustration of the Cartesian product of two sets $S_1 = \{0, 1, 2\}$ and $S_2 = \{0, 1\}$.



Example 3.

Theorem 2. *Let S_1 and S_2 be two finite sets (say, $S_1 = \{0, 1, \ldots, |S_1| - 1\}$ and $S_2 = \{0, 1, \ldots, |S_2| - 1\}$). Then $|S_1 \times S_2| = |S_1||S_2|$.*

Proof. We define a function $f : S_1 \times S_2 \to \{0, 1, \ldots, |S_1||S_2| - 1\}$ as $f((s_1, s_2)) = s_1|S_2| + s_2$ (for all $s_1 \in S_1, s_2 \in S_2$.

First we show that f is injective.

Suppose $f((s_1, s_2)) = f((t_1, t_2))$ for some $s_1, t_1 \in S_1$ and $s_2, t_2 \in S_2$. Then, by definition of f,

$$\begin{aligned} s_1|S_2| + s_2 &= t_1|S_2| + t_2 \\ \left\lfloor \frac{s_1|S_2| + s_2}{|S_2|} \right\rfloor &= \left\lfloor \frac{t_1|S_2| + t_2}{|S_2|} \right\rfloor \end{aligned}$$

But

$$\begin{aligned} \left\lfloor \frac{s_1|S_2| + s_2}{|S_2|} \right\rfloor &= s_1 + \left\lfloor \frac{s_2}{|S_2|} \right\rfloor \\ &= s_1 \end{aligned}$$

since $s_2 \in S_2$ and $S_2 = \{0, \ldots, |S_2| - 1\}$. Similarly

$$\begin{aligned} \left\lfloor \frac{t_1|S_2| + t_2}{|S_2|} \right\rfloor &= t_1 + \left\lfloor \frac{t_2}{|S_2|} \right\rfloor \\ &= t_1 \end{aligned}$$

Therefore, $s_1 = t_1$. Again, from

$$s_1|S_2| + s_2 = t_1|S_2| + t_2$$

we can deduce

$$s_2 = t_2$$

by cancelling $s_1|S_2| = t_1|S_2|$ on both sides.

Therefore, we have shown that if $f((s_1, s_2)) = f((t_1, t_2))$ then $(s_1, s_2) = (t_1, t_2)$. So f is injective.

Second, we show that f is surjective.

Let x be any element of $\{0, 1, \ldots, |S_1||S_2| - 1\}$. We need to find an element of $S_1 \times S_2$ which is assigned (mapped to) k.

Then $\lfloor x/|S_2| \rfloor \in S_1$ and $x \mod |S_2| \in S_2$. And

$$f(\left(\left\lfloor \frac{x}{|S_2|} \right\rfloor, x \mod |S_2|)\right) = \left\lfloor \frac{x}{|S_2|} \right\rfloor |S_2| + (x \mod |S_2|) = x$$

. Since x was an arbitrary element, we have shown that f is surjective.

Thus, f is a bijection and the size of the two sets $S_1 \times S_2$ and $\{0, 1, \ldots, |S_1||S_2| - 1\}$ are equal. $\square$

The above is a (very) formal proof of the intuitive idea that

- we have $|S_1|$ choices for the first coordinate, and
- regardless of our first choice, we have $|S_2|$ choices for the second coordinate.

Of course, we would need to check that different choices do not give us the same element of $S_1 \times S_2$ (injectivity) and we can get every element of $S_1 \times S_2$ (surjectivity).

From now on, we will only give proofs in this less formal format.

Example 4.

Theorem 3. *The complete graph on n vertices (K_n) has $\frac{n(n-1)}{2}$ edges.*

The idea of the proof is that we can choose a vertex u and then we choose another vertex v that is not u in K_n, This will give us every edges of K_n exactly twice (one from each side). There are n choices for the first vertex and $n - 1$ choices for the second vertex so $n(n - 1)$ is twice the number of edges.

Note that in the previous example, we did not use a bijection since the procedure we described gives us every edge exactly twice (so that if we were to define a function, it would not be injective). But it nevertheless allowed us to count.

This idea can be generalized to the following theorem.

Theorem 4. *Let $G = (V, E)$ be a bipartite graph with parts A and B. Then*

$$\sum_{v \in A} \deg(v) = |E| = \sum_{v \in B} \deg(v)$$

Proof sketch 1. Since every edge has one endpoint in A and one endpoint in B, $\sum_{v \in A} \deg(v)$ counts every edge exactly once (namely, from its endpoint in A.

Corollary 1. *Let $f : A \to B$ be a function such that every element of B is a assigned exactly k elements of A. Then $k|B| = |A|$.*

Proof. Consider f as a bipartite graph and apply the previous theorem. ☐

Let us now apply this corollary.

Lemma 1. *The number of subsets of size k of a set of size n is $\binom{n}{k}$.*

Notation.

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

Proof. Let $S = \{s_1, \ldots, s_n\}$.

We can obtain a subset of size k of S by

- picking an element x_1 of S to add to our subset,
- picking an element x_2 of $S \setminus \{x_1\}$ to add to our subset,
- picking an element x_3 of $S \setminus \{x_1, x_2\}$ to add to our subset,
- and so on.

We have n choices for x_1, $n-1$ choice for x_2 and so on. So the total number of choices is $n(n-1)\ldots(n-k+1)$. However, each subset T can be obtained in exactly $k!$ ways. Namely, once for every permutation of the elements of T.

Thus, the number of subsets of size k of S is

$$\frac{n(n-1)\ldots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$$

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