Notes on Hamiltonian cycles

Definition 1. A graph G is *k-connected* if there does not exist a set of at most $k-1$ vertices of G whose removal yield a disconnected graph.

Definition 2. An alternative definition of a *path* is a sequence of vertices of a graph such that consecutive vertices have an edge between them.

The *length* of a path P is the number of vertices in it and is denote $|P|$ or $|V(P)|$. Since a cycle is a path, the *length* of a cycle is the number of vertices in it.

Notation. If G is a graph an S is a subset of the vertices of G then $G - S$ denotes the graph obtained by removing the vertices of S from G. The resulting graph consist of all vertices in $V(G) - S$ and all edges of G between vertices of $V(G) - S$.

Formally, this would be the graph $(V(G) - S, \{uv | uv \in E(G), u \in V(G) - S, v \in V(G) - S\}$.

We usually abuse notation and write $G - x$ to mean $G - \{x\}$ when we want to remove a single vertex x.

Graph theorist also tend to abuse set theoretic notation as we have just done now. Namely, we write $V(G) - S$ to mean $V(G) \setminus S$ (the set $V(G)$ with the element S removed).

Lemma 1. *If a graph G has a Hamiltonian cycle then G is 2-connected.*

Proof. Suppose the lemma is false. Then there exists a graph that is not 2-connected but has Hamiltonian cycle. Let G be such a graph. Then, since G is not 2-connected, there exists a vertex $x \in V(G)$ such that $G - x$ is disconnected. By definition of disconnectivity, there exists two vertices u, v in $G - x$ such that there is no path between u and v in $G - x$.

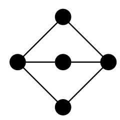
Let $C = c_1, c_2, \ldots, c_k$ be a Hamiltonian cycle of G. u and v appear somewhere in that sequence since C is Hamiltonian. We can relabel u and v so that u appears before v in the sequence. Thus, the subsequence of C from u to v looks like $u, \ldots, v$. x appears in this sequence as otherwise, this subsequence is a path from u to v in $G - x$. Note that the two concatenated subsequences $v, \ldots, c_k, c_1, \ldots, u$ is also a path in G. Thus, x appears in this subsequence as well (as otherwise, we have a path between u and v in $G - x$).

But this means that x appears twice in C. This is a contradiction to the fact that C is a Hamiltonian cycle. □

Note that we do not normally refer to paths and cycles in a graph as sequences. We use the word "subpath" when we want to take a subsequence.

Also note that we would normally say, "without loss of generality, u is before v in the cycle" (which is slightly more ambiguous than saying that we can relabel them).

Now, can we obtain the converse of the previous statement? It happens that the converse of the statement is false. For example, the following graph is 2-connected but does not have a Hamiltonian cycle (check that this is indeed the case).



However, there does exist theorems which give sufficient conditions for the existence of a Hamiltonian cycle.

Theorem 1 (Dirac's theorem)**.** *If a graph G has at least 3 vertices and the degree of every vertex of G is at least $\frac{|V(G)|}{2}$ then G has a Hamiltonian cycle.*

Proof. Suppose the theorem is false. Then there exists a graph G with no Hamiltonian cycle, at least 3 vertices and every vertex of G has degree at least $\frac{n}{2}$ where $n = |V(G)|$.

Let $P = p_1, \ldots, p_k$ be a longest path in G. We claim that G contains a cycle of length k (that is, a cycle with k vertices).

Proof. Suppose p_1 is adjacent to a vertex u of G not on P. Then $u, p_1, \ldots, p_k$ is a longer path than G. This is a contradiction to our choice of P.

Thus, p_1 is only adjacent to vertices of P. By symmetry, p_k is only adjacent to vertices of P.

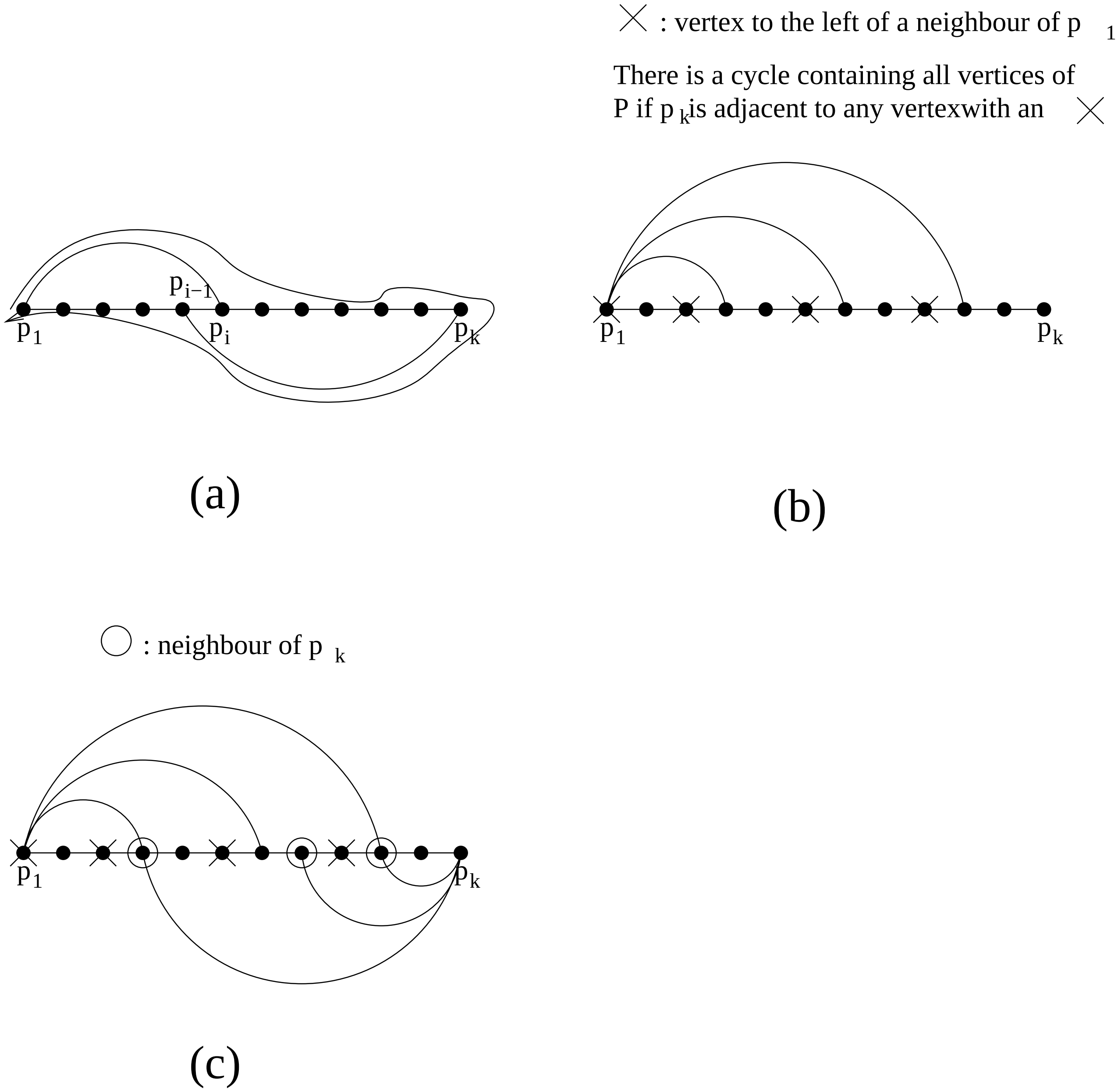


Figure 1: (a) A cycle using the edges p_1p_i and $p_{i-1}p_k$. (b) Identifying all vertices to the left of a neighbour of p_1. (c) Neighbours of p_k. Every vertex in $p_1, \ldots, p_{k-1}$ receives at most one "mark" (X or O) but we need to put at least $\frac{n}{2} + \frac{n}{2}$ marks.

Suppose p_1 is adjacent to p_i and p_k is adjacent to p_{i-1} then $p_1, p_i, p_{i+1}, \ldots, p_{k-1}, p_k, p_{i-1}, \ldots, p_2$ is a cycle of length k.

So we will try to find such an i. If $S = \{i | p_i \in N(p_1)\}$ and $T = \{i+1 | p_i \in N(p_k)\}$ contain a common element then we have found the i we needed. Otherwise, the intersection of S and T is empty and their union is contained in $\{2, 3, \ldots, k\}$. So $k - 1 = |\{2, \ldots, k\}| \geq |S \cup T| = |S| + |T| \geq \frac{n}{2} + \frac{n}{2} = n$. This is a contradiction since $k \leq n$ (there cannot be more vertices in the path P than there are vertices in the whole graph G). $\square$

Let $C = c_1, \ldots, c_k$ be a cycle of length k in G. If there is any edge $c_i u$ between a vertex in C and a vertex not in C then $u, c_i, c_{i+1}, \ldots, c_{k-1}, c_k, c_1, c_2, \ldots, c_{i-1}$ is a path (exercise: check that this is indeed a path). But this path has length $k + 1$.

So there are no edges between vertices in C and vertices not in C. We see that $k \geq n/2 + 1$ since a vertex in C has degree at least $n/2$ and can only have neighbours in C. But similarly, there are at least $n/2 + 1$ not in C since any vertex not in C (and there should be at least one or C is a Hamiltonian cycle) has degree at least $n/2$ and can only have neighbours not in C.

But we need at least $n/2 + 1 + n/2 + 1 = n + 2$ vertices to satisfy both these conditions and G only has n vertices. Contradiction. $\square$