I have shown this example in class but did not get time to prove uniqueness using rules of inference.

Example 1. On a strange island, everyone is either a knight or a knave. Knights always tell the truth and knaves always lie. You mean two people Jack and Tim.

Jack says "If Tim is a knave then I am a knight.". Tim says "We are the different."

What type is each person?

Here is an informally way of answering the question.

If Tim was a knight then Tim is telling the truth so they should be different. Therefore, Jack must be a knave. Now we need to check if this is consistent with what Jack is saying. Tim is not a knave so the premise of the statement Jack is making is false. Thus, the implication is true so Jack is telling the truth. This contradicts the deduction we have just made stating that Jack is a knave.

On the other hand if Tim was a knave then he is lying. So they are not different. Therefore, Jack is also a knave. Again, we check if this is consistent with what Jack is saying. The premise of his statement is true (Tim is indeed a knave) but the conclusion is false (Jack is not a knight). Thus the implication is false (in fact, this is the only way for an implication to be false). So Jack is indeed lying as we expected him to.

Note that by the second sentence of the previous paragraph, this informal argument has shown that the only possibility that is left is that Tim and Jack are both knaves.

We then verified that this solution is consistent with their statements. However, in general, this technique **does not work**. It works in this case because we know all statements we could possibly need to check. Normally, checking that a statement does not lead to a contradiction **does not consist of a proof** of the statement (this is a very common mistake!).

Proof. Formally, we let p be the proposition "Jack is a knight" and q be the proposition "Tim is a knight". Then Jack's statement is "$\neg q \rightarrow p$".

p	q	$\neg q \rightarrow p$	$(\neg p \wedge q) \vee (p \wedge \neg q)$	$p \leftrightarrow (\neg q \rightarrow p)$	$q \leftrightarrow ((\neg p \wedge q) \vee (p \wedge \neg q))$
T	T	T	F	T	F
T	F	T	T	T	F
F	T	T	T	F	T
F	F	F	F	T	T

If we wanted to prove that there is at most one solution, namely $\neg p \wedge \neg q$ then we proceed as follows.

1	$p \leftrightarrow (\neg q \rightarrow p)$	premise
2	$q \leftrightarrow ((\neg p \wedge q) \vee (p \wedge \neg q))$	premise
3	q	assumption
4	$\neg q$	assumption
5	$\mathbf{F}$	$3, 4, \neg\mathcal{E}$
6	p	$5, \mathbf{F}\mathcal{E}$
7	$\neg q \rightarrow p$	$4-6, \rightarrow\mathcal{I}$
8	$(\neg p \wedge q) \vee (p \wedge \neg q)$	$2, 3,$ special$\leftrightarrow\mathcal{E}$
9	$(p \wedge \neg q)$	assumption
10	$\neg q$	$9, \wedge\mathcal{E}$
11	$\mathbf{F}$	$3, 10, \neg\mathcal{E}$
12	$(p \wedge \neg q) \rightarrow \mathbf{F}$	$5-7, \rightarrow\mathcal{I}$
13	$(\neg p \wedge q)$	assumption
14	$\neg p$	$13, \wedge\mathcal{E}$
15	p	$1, 7, \rightarrow\mathcal{E}$
16	$\mathbf{F}$	$14, 15, \neg\mathcal{E}$
17	$(\neg p \wedge q) \rightarrow \mathbf{F}$	$13-16, \rightarrow\mathcal{I}$
18	$\mathbf{F}$	$8, 12, 17, \vee\mathcal{E}$
19	$q \rightarrow \mathbf{F}$	$3-18, \rightarrow\mathcal{I}$
20	$\neg q$	$19, \neg\mathcal{I}$
21	$\neg((\neg p \wedge q) \vee (p \wedge \neg q))$	$2, 20,$ special$\leftrightarrow\mathcal{E}$
22	p	assumption
23	$p \wedge \neg q$	$20, 22, \wedge\mathcal{I}$
24	$(\neg p \wedge q) \vee (p \wedge \neg q)$	$22, \wedge\mathcal{I}$
25	$\mathbf{F}$	$21, 23, \neg\mathcal{E}$
26	$p \rightarrow \mathbf{F}$	$21-24, \rightarrow\mathcal{I}$
27	$\neg p$	$25, \neg\mathcal{I}$
28	$\neg p \wedge \neg q$	$20, 26, \wedge\mathcal{I}$

Again, this only shows that the only possible solution is that Jack and Tim are both knaves. It could be that there are no solution. Although, in this particular case, we would expect there to be at least one solution (how could they be of no type?) □