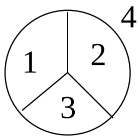
Note on planar graphs and colouring

The four colour theorem states that any map can be coloured with four colours so that regions sharing a boundary receive the same colour. It is easy to see that four colours are necessary since all regions share a boundary in the following example.



The theorem can be formulated in graph theoretic terms as a problem about colouring the vertices of a "planar graph".

Definition 1. A graph is *planar* if it can be drawn in the plan in such a way that no two edges cross.

Note that it is possible to draw the same planar graph in different ways. For example, here are two drawings of the same graph.

Definition 2. A planar graph with a drawing is called a *plane graph*.
In a plane graph, the plane is split into regions which we call *faces*
There is always exactly one "infinite" face which we call the *outer face*.

Notation. $F(G)$ denotes the faces of a plane graph G.
For a face $f \in F(G)$, $bd(f)$ is the *boundary of* f. $|bd(f)|$ is the number of edges in the boundary.

Note that all definitions extend to multigraphs.
We can now state the following well known theorem. Although, usually, it is stated in terms of polyhedra.

Theorem 1. *If G is a connected plane (multi)graph then*

$$|V(G)| + |F(G)| = |E(G)| + 2$$

Proof. Suppose the theorem is false. Let G be a minimum counter-example minimizing $|E(G)|$.
If G is a tree then by lemma, $|E(G)| = |V(G)| - 1$. Further, G has only one face. Therefore

$$|E(G)| + 2 = |V(G)| - 1 + 2 = |V(G)| + 1 = |V(G)| + |F(G)|$$

. Contradiction to G being a counter-example.
If G is not a tree then G contains a cycle C. Let e be any edge in C.
$G - e$ is connected since any path using e can be replaced by a path using the other edges of C.
$G - e$ is a plane graph since we can keep the same drawing for G and delete the line corresponding to e.
Therefore, by minimality,

$$|V(G - e)| + |F(G - e)| = |E(G - e)| + 2$$

Since $G - e$ is only one less edge than G,

$$|V(G - e)| = |V(G)||E(G - e)| = |E(G)| - 1$$

IF e is in the boundary of two different faces (one on each "side" of the edge) then those two faces are merged into a single face in $G - e$. So $|F(G - e)| + 1 = |F(G)|$ in this case.
Actually, e cannot be in the boundary of two faces. This is since one face muse be on the inside of C and the other is on the outside of C.
Substituting these $|V(G - e)|, |E(G - e)|$ and $|F(G - e)|$ in

$$|V(G - e)| + |F(G - e)| = |E(G - e)| + 2$$

gives

$$|V(G)| + |F(G)| - 1 = |E(G)| - 1 + 2$$

and

$$|V(G)| + |F(G)| = |E(G)| + 2$$

. This is a contradiction to G being a counter-example. $\square$

Corollary 1. *If G is a connected (simple) plane graph with at least 2 edges then*

$$|E(G)| \leq 3|V(G)| - 6$$

Proof. In a simple connected plane graph G with at least 2 edges, every face boundary contains at least 3 edges. Otherwise, a face boundary would only contain 2 vertices which results in parallel edges.

Since every edge appears exactly twice in all face boundaries,

$$2|E| = \sum_{f \in F(G)} |bd(f)| \leq \sum_{f \in F(G)} 3 = 3|F| \leq 3(|E(G)| - |V(G)| + 2)$$

. The second inequality is obtained using Euler's formula. Rearranging gives

$$3|V(G)| - 6 \geq 2|E(G)|$$

. $\square$

Definition 3. A *colouring* with k colours (or *k-colouring*) of a graph $G = (V, E)$ is an assignment $c : V \to \{1, 2, \ldots, k\}$ such that adjacent vertices are assigned different values. I.e., $\forall e = (u, v) \in E, c(u) \neq c(v)$

If there is a k-colouring of G then we say that G is *k-colourable*.

Theorem 2. *Every planar graph is 6-colourable.*

Proof. Suppose the theorem is false. Let G be a counter-example minimizing $|V(G)|$.

$V(G)$ is not empty since the empty assignment is a 6-colouring.

By Corollary 1,

$$6|V(G)| - 12 \geq 2|E(G)| = \sum_{v \in V(G)} \deg(v)$$

so if we divide by $|V(G)|$ on both sides,

$$6 - \frac{12}{|V(G)|} = \frac{1}{|V(G)|} \sum_{v \in V(G)} \deg(v)$$

. Therefore, the average degree is G is less than 6. So there exists a vertex v in G of degree at most 5.

By minimality of G, $G - v$ has a 6-colouring c.

Since $\deg(v) \leq 5$, some colour in $\{1, 2, 3, 4, 5, 6\}$ does not appear on $N(v)$ in c. Thus, we can extend c to a 6-colouring of G by setting $c(v)$ to the missing colour.

Contradiction to G being a counter-example. $\square$

Theorem 3. *Every planar graph is 5-colourable.*

Proof. Suppose the theorem is false. Let G be a counter-example minimizing $|V(G)|$.

$V(G)$ is not empty since the empty assignment is a 6-colouring.

By Corollary 1, G contains a vertex v of degree at most 5.

By minimality, $G - v$ has a 5-colouring c.

If v has degree at most 4 then with c is missing a colour on $N(v)$. We can extend c this colouring to G by colouring v with the missing colour. Contradiction to G being a counter-example.

Similarly, if $N(v)$ is missing a colour then we can extend c this colouring to G by colouring v with the missing colour. Contradiction to G being a counter-example.

Thus, $N(v) = 5$ and all 5 colours appear on $N(v)$.

We label the vertices around v in clockwise order by v_1, v_2, v_3, v_4, v_5. We relabel the colours so $c(v_i) = i$.

We now try to find a colouring of $G - v$ where some colour is missing on $N(v)$.

Let $G_{1,3}$ be the induced subgraph of $G - v$ with vertex set all vertices coloured 1 or 3 by c (and all edges of G between these vertices).

If the connected component C of $G_{1,3}$ containing v_1 does not contain v_3, swap colours 1 and 3 on all vertices in C. I.e., c' is a 5-colour of G where

- $c'(v) = 1$,
- $c'(u) = 1$ if $u \in C$ and $c(u) = 3$,
- $c'(u) = 3$ if $u \in C$ and $c(u) = 1$,
- $c'(u) = c(u)$ otherwise

If C does contain v_3 then let $G_{2,4}$ be the induced subgraph of $G - v$ with vertex set all vertices coloured 2 or 4 by c. If the connected component D of $G_{2,4}$ containing v_2 does not contain v_4, swap colours 2 and 4 on all vertices in D. I.e., c' is a 5-colour of G where

- $c'(v) = 2$,
- $c'(u) = 2$ if $u \in C$ and $c(u) = 4$,
- $c'(u) = 4$ if $u \in C$ and $c(u) = 2$,
- $c'(u) = c(u)$ otherwise

In fact, there cannot be a path from v_1 to v_3 in $G_{1,3}$ and a path from v_2 to v_4 in $G_{2,4}$. Otherwise, these path would cross. They cannot cross at a vertex since $G_{1,3}$ and $G_{2,4}$ contain different vertices. So these path croess at an edge. Contradiction. $\square$

Actually, the last part of the above proof uses what is known as the Jordan curve theorem. It eseentially states that to go from the "inside" of a closed curve (here, it is the cycle created by P_1 and v) to the outside, we need to intersect the curve.