1 Edmonds matching polytope

The first LP one might think of for describing the matching problem is:

(MP)

$$\begin{aligned} \text{maximise} \quad & \sum_{e \in E(G)} x_e \\ \text{subject to} \quad & \sum_{u \in N(v)} x_{uv} \leq 1 \ \forall v \in V(G) \\ & 0 \leq x_e \leq 1 \qquad \forall e \in E(G) \end{aligned}$$

Theorem 1.1. *For a bipartite graph G, (MP) is integral. That is any fractional solution is a convex combination of integral solution.*

Proof. Suppose not and let $\vec{x}$ be an optimal solution to (MP) with minimum number of fractional entries $0 < x_e < 1$.

If the subgraph of fractional valued edges contains a cycle C, we can alternate between adding $+\varepsilon$ and $-\varepsilon$ on the edges of C. This gives us two new solutions: one for the maximum ε which keeps the solution feasible and one for the minimum ε (which is negative). Now $\vec{x}$ is a convex combination of these two solutions, each with fewer fractional entries. By minimality, they are both convex combinations of integer vectors, but then so is $\vec{x}$ (as a convex combination of a convex combination is a convex combination).

So the subgraph of fractional valued edges is a forest. For any path between two leaves of the same tree of this forest, we can again alternate between adding and subtracting ε on this path to show $\vec{x}$ is a convex combination, a contradiction.

□

We did not use the objective in our proof! The result is purely about polytopes.

So with an algorithm to solve LPs, we could find the optimum for any objective function, not just the all ones vector. I.e., we can find a maximum weight matching in a bipartite graph.

For the general case, triangles are a problem. The vector with value $\frac{1}{2}$ for all edges isn't convex combination of anything (except itself).

Thus, for general graph, we need to consider the *Edmonds' matching polytope.*

(EMP)

$$\begin{aligned}
\text{maximise} \quad & \sum_{e\in E(G)} x_e \\
\text{subject to} \quad & \sum_{u\in N(v)} x_{uv} \le 1 && \forall v \in V(G) \\
& \sum_{e\in E(G[S])} x_e \le \frac{|S|-1}{2} && \forall S \subseteq V, |S| \text{ odd} \\
& 0 \le x_e \le 1 && \forall e \in E(G)
\end{aligned}$$

Theorem 1.2. *For any graph G, the vertices of (EMP) is integral.*

Let's step back and prove the theorem for perfect matchings first. I.e., the polytope defined by

(PMP)

$$\begin{aligned}
& \sum_{u\in N(v)} x_{uv} = 1 \ \forall v \in V(G) \\
& \sum_{e\in \delta(S)} x_e \ge 1 \quad \forall S \subseteq V, |S| \ge 3, |V-S| \ge 3 \text{ odd} \\
& 0 \le x_e \le 1 \qquad \forall e \in E(G)
\end{aligned}$$

Notation: $\delta(S)$ is all edges out of S.

Theorem 1.3. *For any graph with an even number of vertices, the vertices of (PMP) are integral.*

A graph with an odd number has no perfect matchings so the polytope describing all solutions would be empty.

Proof. Suppose not and let $G, \vec{x}$ be a counter-example minimizing G and subject to this minimizes the number of fractional entries, and subject to both of these minimize the size of the support graph of $\vec{x}$.

In fact, we can assume the support is G or otherwise, the support graph is also a counter-example. Furthermore, we can suppose (for a contradiction) that $\vec{x}$ is a vertex of the polytope.

As before G contains no even cycle. Since we are looking at the perfect matching polytope, it has no degree 1 vertex either.

So all vertices have degree at least 2. If all vertices have degree exactly 2, G is the disjoint union of cycles. So all cycles are odd or we use the same argument as the even cycle case. Take one such cycle C. Now by the odd set constraint, at least two vertices on C do not satisfy the first constraint with equality.

So all vertices have degree 2 and some vertex has degree at least 3.

Since $\vec{x}$ is a vertex and it has $|E(G)|$ variables, it satisfies at least $|E(G)| > |V(G)|$ inequalities with equality. So at least one odd set inequality with equality, for say S.

Now, attempt to contract each side S and $G - S$ and apply minimality.

Let $x^{G/S}$ be the corresponding solution for G/S (keep duplicate edges, but not loops). By minimality, it is convex combination of integer vectors.

Let $x^{G/(V-S)}$ be the corresponding solution for $G/(V - S)$ (keep duplicate edges, but not loops). By minimality, it is convex combination of integer vectors.

Integer vectors are perfect matchings. By taking the least common multiple k of all denominators for coefficients of the convex combinations, we can assume both $x^{G/S}$ and $x^{G/(V-S)}$ are convex comb of the same number of matchings (with possibly multiple copies of a matching). I.e., there's a multiset $\mathcal{M}_1$ of matchings of G/S and a multiset $\mathcal{M}_2$ of matchings of $G/(V - S)$ such that

$$x^{G/S} = \frac{1}{k} \sum_{M \in \mathcal{M}_1} \mathbf{1}_M$$
$$x^{G/(V-S)} = \frac{1}{k} \sum_{M \in \mathcal{M}_2} \mathbf{1}_M.$$

where $\mathbf{1}_M$ denotes the vector that is 1 for each index e that is an edge in M and 0 everywhere else. This is called the *characteristic vector of a matching*.

Because of the value $\vec{x}$ takes, for each $e \in \delta(S)$, the total number matchings containing e in $\mathcal{M}_1$ and $\mathcal{M}_2$ are the same.

So we can pair them up the matchings along each edge e of $\delta(S)$ they contain and take the union of the two matchings in each pair to form matchings of G. Then, $\vec{x}$ is $\frac{1}{k}$ times the sum characteristic vectors of these matchings. I.e., $\vec{x}$ is a convex combination of integeral vectors, a contradiction.

□

Now let's prove the integrality for (EMP) for non-perfect matchings.

Proof. We will reduce to the perfect matching case. Build an auxiliary graph G' by making two copies of G and adding an edge between the two copies of the same vertex to form a new graph G'.

We just need to show the corresponding vector $\vec{x}'$, defined as being equal to $\vec{x}$ for entries inside each copy and equal to the remainder on a vertex for an edge between copies, satisfies both set of constraints. The first set of (equality) are satisfied by construction (the weight of edges between copies are chosen precisely to satisfy those constraints). It remains to show.

$$\sum_{e \in \delta(S)} x'_e \geq 1 \ \forall S \subseteq V, |S| \text{ odd}, |S| \geq 3, |V - S| \geq 3$$

Then the convex combination we get for $\vec{x}'$ gives us corresponding convex combination in the original graph.

To show this constraint is satisfied for any such $S \subseteq V(G')$, we note that S is the union of S_1 in the first copy and S_2 in the second copy. This partitions G' into 4 sets and there are four types of edges in $\delta(S)$ as shown here.

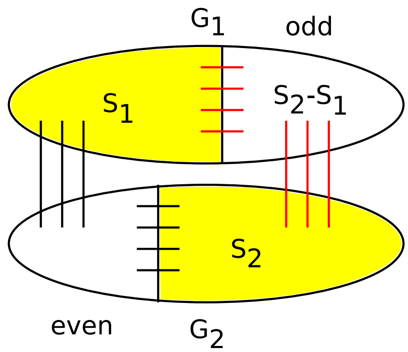

Figure 1: Edges in $\delta(S)$. The sum of across red edges is at least 1 under these parity assumptions.

Since $|S|$ is odd S_1 and S_2 of different parity. Without loss of generality, $|S_1|$ is odd.

Since $|S|$ is odd S_1 and S_2 of different parity and so $S_1 - S_2$ and $S_2 - S_1$ are of different parity. Without loss of generality, $|S_2 - S_1|$ is odd. Combining the second inequality of (EMP) for $S = S_2 - S_1$ and the first inequality of (PMP) for all vertices in $S_2 - S_1$, we see that there's at least total weight 1 in $\delta(S_2 - S_1)$. But we have all these edges in $\delta(S)$ and therefore the second constraint of (PMP) is satisfied.

Now simply restrict the convex combination of $\vec{x}'$ in terms of integer vectors to the edges of, says, the first copy of G. This shows $\vec{x}$ is convex combination of integer vectors.

□